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LOCALIZATION IN HYPERELASTO-PLASTIC POROUS SOLIDS SUBJECTED TO UNDRAINED CONDITIONS

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Abstract—The paper analyses the conditions for a developing localization zone in a porous solid (soil) subjected to undrained conditions, while the solid skeleton is assumed to undergo finite elastic–plastic deformation. A basic feature of the analysis is the concept of a regularized displacement discontinuity (and the consequent pore pressure discontinuity) which was recently adopted by Larsson *et al.* (1996) in the context of geometrically linear theory. The geometric effect due to elastic compressive strains is investigated in a numerical example of biaxial loading under the plane strain conditions. The findings might have relevance for the interpretation of phenomena involving soft and weak geomaterials. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The development of localized failure zones in porous materials, such as soils, has attracted considerable attention in recent years from theoretical as well as experimental viewpoints. Most theoretical analyses have considered pressure-dependent behavior under drained conditions, whereby the interaction with the pore fluid is disregarded. Important contributions include, among others, Mandel (1964), Rudnicki and Rice (1975), Rice and Rudnicki (1980). In practice, however, the interaction between the solid skeleton and the pore fluid will have a significant influence on the formation of localization zones, its orientation, as well as the (critical) level at which localization becomes possible. In particular, the extreme situation of (locally or globally) undrained behavior, in which case the net drainage of pore fluid from or to a given spatial volume is prevented, has attracted attention, cf Rice and Cleary (1976), Rudnicki (1983), Han and Vardoulakis (1991), Runesson *et al.* (1996), Larsson *et al.* (1996). In the latter reference, a finite element analysis scheme was proposed for capturing the developing displacement and pore pressure discontinuities across the anticipated localization band: the "embedded localization band" approach. It is emphasized that the locally undrained condition represents a quite extreme case, which may be difficult to achieve in practice, except for the very onset of localization and at loading rates much higher than the dissipation rate in the region of the forming localization band. For example, experiments carried out by Desrue *et al.* (1993) show that shear band development induces a significant increase of drainage capacity (in the postlocalized regime) within the band.

When geometrically nonlinear effects have been considered, they have been treated within the context of hypoelastic–plastic response, which does not have a firm thermodynamic basis. Thus, in this paper we adopt the hyperelastic–plastic constitutive assumption for application to a porous solid skeleton. Hence, the paper may be viewed, on one hand, as a natural extension of the developments by Steinmann *et al.* (1996) regarding large

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strain localization in one-phase solids. On the other hand, the paper may be viewed as an extension of Runesson *et al.* (1996) regarding undrained analysis under the assumption of small strains. It may be worth mentioning that the analysis of undrained behavior resembles that of adiabatic response in thermoelastic–plastic solids, whereby the temperature plays the role of pore pressure e.g. Benallal *et al.* (1992), Runesson *et al.* (1996).

The outline of the paper is as follows. First we give a brief review of the basic field equations for the mixture of solid and pore fluid in a spatial setting. Assuming a hyperelastic–plastic solid skeleton and a logarithmic law for pore pressure in terms of the pore fluid volume change, we derive the tangent stiffness relations for the porous medium subjected to undrained conditions. We then proceed to the localization analysis, which is based on the concept of a regularized displacement discontinuity, which defines the anticipated localization band. Restricting our discussion to the particular conditions at the onset of localization, we consider three possible scenarios with respect to the possible combinations of plastic loading/unloading inside and outside of the anticipated band. It appears not to be possible to arrive at closed-form expressions for the critical orientation of the localization band (except in the special case of small elastic strains). Finally, numerical results are presented for plane strain, whereby the classical Mohr–Coulomb yield criterion (in terms of Kirchhoff stress) is adopted in conjunction with the Neo-Hooke hyperelastic model.

2. FIELD EQUATIONS FOR SOIL WITH PORE FLUID

Let $\boldsymbol{\sigma}$ denote the solid or "effective" Cauchy stress, and p denote the excess pore fluid pressure in the mixture of solid and fluid.[1] The equilibrium equation can be formulated in the spatial format as

$$-\boldsymbol{\nabla}_x \cdot \bar{\boldsymbol{\sigma}} = \boldsymbol{b} \quad \text{with } \bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - p\boldsymbol{\delta} \tag{1}$$

where $\bar{\boldsymbol{\sigma}}$ is the total stress, $\boldsymbol{b}$ is the body force per unit bulk-volume, and $\boldsymbol{\delta}$ is the spatial second-order identity tensor.[2] The rate form of (1) is

$$-\boldsymbol{\nabla}_x \cdot \mathring{\bar{\boldsymbol{\sigma}}} = \dot{\boldsymbol{b}} \quad \text{with } \mathring{\bar{\boldsymbol{\sigma}}} = \mathring{\boldsymbol{\sigma}} - \dot{p}\boldsymbol{\delta} - p[(\mathrm{tr}\,\boldsymbol{d})\boldsymbol{\delta} - \boldsymbol{l}] \tag{2}$$

where $\boldsymbol{l}$ is the velocity gradient ($\boldsymbol{l} = \boldsymbol{v} \otimes \boldsymbol{\nabla}_x$, $\boldsymbol{v}$ is the spatial velocity), whereas $\mathring{\boldsymbol{\sigma}}$ is the Truesdell rate of $\boldsymbol{\sigma}$ defined as

$$\mathring{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l} \cdot \boldsymbol{\sigma} + (\boldsymbol{\sigma} \otimes \boldsymbol{\delta}) : \boldsymbol{l} \tag{3}$$

Besides equilibrium, it is necessary to establish the continuity, or mass balance, equation for the mixture of solid and fluid. In the spatial format, this equation takes the form

$$\boldsymbol{\nabla}_x \cdot \boldsymbol{g} = -\mathrm{tr}\,\boldsymbol{d} + d^{\mathrm{f}}_{\mathrm{vol}} \tag{4}$$

where $\boldsymbol{g}$ is the drainage flux of fluid, i.e. basically the velocity of fluid relative to the solid skeleton, and $d^{\mathrm{f}}_{\mathrm{vol}}$ is the spatial rate of volume change of the pore fluid.[3] The special case of undrained condition is defined as

$$\boldsymbol{\nabla}_x \cdot \boldsymbol{g} = 0 \rightsquigarrow \mathrm{d}^{\mathrm{f}}_{\mathrm{vol}} = \mathrm{tr}\,\boldsymbol{d} \tag{5}$$

[1] Here we used "the principle of effective stress" cf Terzaghi and Peck (1948).

[2] Energy arguments that $\boldsymbol{\sigma}$ and p should be used to represent solid and fluid phase constitutive behavior rather than partial stresses (from mixture theory) were given by Biot (1941); see also Borja and Alarcon (1995).

[3] It is possible, but not necessary, to think of $d^{\mathrm{f}}_{\mathrm{vol}}$ as $\mathrm{tr}\,\mathbf{d}^{\mathrm{f}}$, where $\mathbf{d}^{\mathrm{f}} = (\boldsymbol{v}^{\mathrm{f}} \otimes \boldsymbol{\nabla}_x)^{\mathrm{sym}}$ is the symmetric part of the spatial velocity gradient of the pore fluid.

3. CONSTITUTIVE RELATIONS

3.1. *Constitutive assumptions for the solid phase*

The tangent stiffness relation associated with $\mathring{\boldsymbol{\sigma}}^{\mathrm{t}}$ must be established. To this end, we first postulate a generic relation for $\hat{\boldsymbol{\tau}}^{\mathrm{cc}}$ (the Oldroyd rate of $\boldsymbol{\tau}$) of the form

$$\hat{\boldsymbol{\tau}}^{\mathrm{cc}} = \mathscr{E}_2 : \boldsymbol{d} \tag{6}$$

where $\mathscr{E}_2$ is the second spatial tangent tensor. (Below, we briefly review the developments leading to (6) for a hyperelastic–plastic model. For a more detailed study the reader is referred to Miehe (1992) and Steinmann *et al.* (1996).)

Next, we introduce the nominal (nonsymmetric) stress rate $(\hat{\boldsymbol{\tau}}^{\mathrm{c}})^{\mathrm{t}}$ and its relation to $\mathring{\boldsymbol{\sigma}}^{\mathrm{t}}$:

$$(\hat{\boldsymbol{\tau}}^{\mathrm{c}*})^{\mathrm{t}} = \hat{\boldsymbol{\tau}}^{\mathrm{cc}} + \boldsymbol{\tau}\cdot\boldsymbol{l}, \quad \mathring{\boldsymbol{\sigma}}^{\mathrm{t}} = J^{-1}(\hat{\boldsymbol{\tau}}^{\mathrm{c}*})^{\mathrm{t}} \tag{7}$$

By combining (6) with $(7)_1$, we obtain the relation

$$(\hat{\boldsymbol{\tau}}^{\mathrm{c}*})^{\mathrm{t}} = \mathscr{E}_1 : \boldsymbol{l} \quad \text{with}\, \mathscr{E}_1 = \mathscr{E}_2 + \boldsymbol{\delta}\,\bar{\otimes}\,\boldsymbol{\tau} \tag{8}$$

whereby it is noted that the first spatial tangent tensor $\mathscr{E}_1$ does not possess minor symmetry.

In this paper, we shall assume hyperelastic–plastic response together with non-associative flow and hardening rules. However, we restrict our discussion to isotropic elasticity and plasticity (in terms of initial isotropy as well as isotropic hardening). Due to the (assumed) elastic and plastic isotropy, it is natural to express the yield function as $\Phi(\boldsymbol{\tau}, K_\alpha)$ and the plastic potential (governing the flow and hardening rules) as $\Phi^*(\boldsymbol{\tau}, K_\alpha)$, where K_α are the dissipative (hardening) stresses. It is then possible to establish the tangent operator $\mathscr{E}_2^{\mathrm{ep}}$, pertinent to plastic loading, as

$$\mathscr{E}_2^{\mathrm{ep}} = \mathscr{E}_2^{\mathrm{e}} - \frac{1}{h}(\mathscr{E}_2^{\mathrm{e}} : \boldsymbol{m}^* + 2\boldsymbol{m}^*\cdot\boldsymbol{\tau}) \otimes (2\boldsymbol{m}\cdot\boldsymbol{\tau} + \boldsymbol{m} : \mathscr{E}_2^{\mathrm{e}}) \tag{9}$$

where we have introduced the auxiliary notation

$$h = \boldsymbol{m} : \mathscr{E}_2^{\mathrm{e}} : \boldsymbol{m}^* + 2\,\mathrm{tr}(\boldsymbol{m} : \boldsymbol{\tau} : \boldsymbol{m}^*) + H, \quad \boldsymbol{m} \overset{\mathrm{def}}{=} \frac{\partial \Phi}{\partial \boldsymbol{\tau}}, \quad \mathbf{m}^* \overset{\mathrm{def}}{=} \frac{\partial \Phi^*}{\partial \boldsymbol{\tau}} \tag{10}$$

with H being a suitably defined hardening modulus. Moreover, we introduced the elastic tangent operator $\mathscr{E}_2^{\mathrm{e}}$, pertinent to elastic unloading. The criterion for plastic loading (L) is obtained as

$$(\boldsymbol{m} : \mathscr{E}_2^{\mathrm{e}} + 2\boldsymbol{m}\cdot\boldsymbol{\tau}) : \boldsymbol{d} > 0 \tag{11}$$

This completes a brief description of the hyper-elastic response.

3.2. *Constitutive assumptions for the fluid phase*

As to the motion of fluid through the solid skeleton and the pore fluid deformation, the following constitutive laws are adopted in the spatial format:

$$\boldsymbol{g} = -\boldsymbol{K}\cdot\boldsymbol{\nabla}_x p, \quad d_{\mathrm{vol}}^{\mathrm{f}} = -\frac{1}{K_f}\dot{p} \tag{12}$$

where $\boldsymbol{K}$ is the Darcy permeability tensor, whereas K_{f} is the bulk stiffness modulus of the pore fluid. In general, both $\boldsymbol{K}$ and K_{f} are state-dependent. Upon combining (12) with the continuity eqn (4), we obtain

$$\operatorname{tr} \boldsymbol{d} = -\nabla_X \cdot (\boldsymbol{K} \cdot \nabla_X p) - \frac{1}{K_f} \dot{p} \tag{13}$$

Henceforth, we shall consider only the undrained behavior (as a result of rapid loading or vanishing permeability), in which case (13) gives the simple relation

$$\dot{p} = -K_f \operatorname{tr} \boldsymbol{d} \tag{14}$$

In the particular case that K_f is constant, it is possible to integrate (14), while using the identity $\operatorname{tr} \boldsymbol{d} = J^{-1} \dot{v}$, to obtain the "logarithmic law"

$$p = -K_f \ln J \tag{15}$$

cf the logarithmic elastic law by Weber and Anand (1990).

3.3. *Constitutive assumptions for bulk behavior at undrained conditions*

We are now in the position to establish the "undrained tangent tensor" $\mathscr{E}_1^u$ in the spatial format, defined via the relation

$$(\hat{\bar{\tau}}^{c*})^t = \mathscr{E}_1^u : \boldsymbol{l} \quad \text{with } \hat{\bar{\tau}}^{c*} = J\overset{\circ}{\bar{\boldsymbol{\sigma}}} \tag{16}$$

Upon combining $(16)_2$ with $(2)_2$, $(7)_2$ and (8), we obtain

$$\mathscr{E}_1^u = \mathscr{E}_2^u + \boldsymbol{\delta} \bar{\otimes} \bar{\boldsymbol{\tau}} \quad \text{with } \mathscr{E}_2^u = \mathscr{E}_2 + JK_f \boldsymbol{\delta} \otimes \boldsymbol{\delta} + Jp(2I^{\text{sym}} - \boldsymbol{\delta} \otimes \boldsymbol{\delta}) \tag{17}$$

where I^{sym} is the (symmetric) fourth-order identity tensor. By comparison with the results by Runesson *et al.* (1996), it appears that the first two terms of $\mathscr{E}_1^u$ in (17) are present in the geometrically linear theory (with due adjustment), whereas the last two terms stem entirely from configuration changes and vanish in the case of small deformations.

4. STRONG DISCONTINUITY ALONG MATERIAL SURFACE

4.1. *Preliminaries*

We shall pursue the analysis along the lines laid out by Steinmann *et al.* (1996) for the one-phase material. Hence, we consider a solid body in the material configuration Ω_0, as shown in Fig. 1. Below the load level at which localization is possible, it is assumed that the deformation map $\boldsymbol{\varphi}(\boldsymbol{X}, t)$ is spatially continuous. As a postlocalized state, $\boldsymbol{\varphi}(\boldsymbol{X}, t)$ continues to be smooth, except across the material surface Γ_0 defined as

$$\Gamma_0 = \{\boldsymbol{X} \in \Omega_0 \mid S(\boldsymbol{X}) = 0\} \tag{18}$$

where we have introduced the monotonic function $S(\boldsymbol{X}) : \mathbb{R}^3 \to \mathbb{R}$ on Ω_0. The material surface Γ_0 subdivides Ω_0 into two subdomains Ω_0^- and Ω_0^+, which are defined by $S(\boldsymbol{X}) < 0$ and $S(\boldsymbol{X}) > 0$, respectively. Moreover, Γ_0 has the unit normal $\mathbf{N}$ defined as[4]

$$\mathbf{N} = \nabla_X S \tag{19}$$

and it follows readily that $\boldsymbol{N}$ points from Ω_0^- to Ω_0^+, which is also shown in Fig. 1.

During the deformation $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$, the surface Γ_0 is convected to the spatial surface Γ defined as

[4] Without loss of generality, it is assumed that $|\nabla_X S| = 1$, by which $\boldsymbol{N}$ is a normal vector.

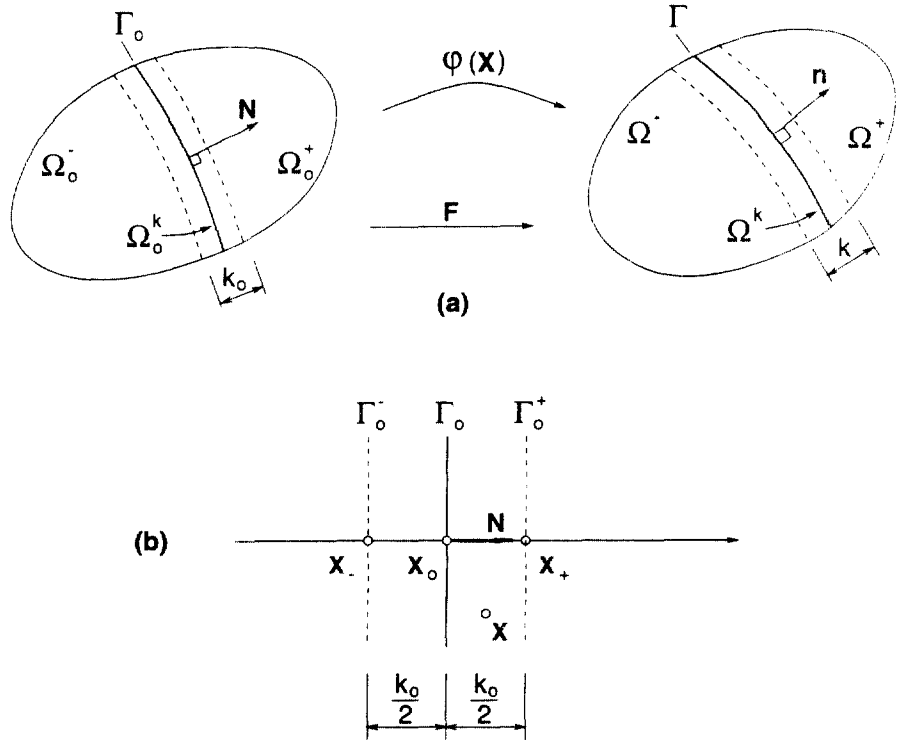

Fig. 1. (a) Solid with regularized singular surface in material and spatial format. (b) Band-shaped zone.

$$\Gamma = \{x \in \Omega \mid s(x, t) = 0\} \quad \text{with } s(x, t) \overset{\text{def}}{=} S(\varphi^{-1}(x, t)) \tag{20}$$

where Ω is the current configuration. The normal n of Γ is given as

$$n = j^{-1}\nabla_x s = j^{-1} N \cdot F^{-1} \quad \text{with } j = |\nabla_x s| \tag{21}$$

The purpose of the subsequent developments is to establish the necessary conditions for the existence of a strong discontinuity in the displacement field across Γ_0. To this end, we first introduce the function $\mathscr{H}_S$ and distribution δ_S in Ω_0 via the standard Heaviside function $\mathscr{H}(y)$ and Dirac-delta distribution $\delta(y) = \mathrm{d}\mathscr{H}(y)/\mathrm{d}y$ as follows

$$\mathscr{H}_S(X) = \mathscr{H}(S(X)) = \begin{cases} 0 & \text{if } X \in \Omega_0^- \\ \frac{1}{2} & \text{if } X \in \Gamma_0 \\ 1 & \text{if } X \in \Omega_0^+ \end{cases} \quad \text{and} \quad \delta_S(X) = \delta(S(X)) \tag{22}$$

We also define the vector-valued Dirac-delta $\boldsymbol{\delta}_S(X)$ formally by the functional identity

$$\boldsymbol{\delta}_S(X) = \nabla_X \mathscr{H}_S(X) \tag{23}$$

in the distributional sense.

Returning to the assumption that $\varphi(X, t)$ may be (spatially) discontinuous across Γ_0, we propose that the displacement field $u(X, t)$ is decomposed into a continuous part $u_c(X, t)$ and another part containing a strong discontinuity, i.e.

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{u}_c(\boldsymbol{X}, t) - [\![\boldsymbol{u}(t)]\!]\mathscr{H}_S(\boldsymbol{X}) \tag{24}$$

where $[\![\boldsymbol{u}(t)]\!]$ is the jump of $\boldsymbol{u}$ across Γ_0 which is assumed to be spatially constant along Γ_0. To simplify the notation, we shall henceforth omit the arguments of field quantities, when there is no risk of confusion.

4.2. *Regularized discontinuity*

Next, we introduce regularized forms of $\mathscr{H}_S$ and δ_S in a fashion that is similar to geometrically linear theory. To this end, we assume that two parallel surfaces Γ_0^- and Γ_0^+ with the (same) unit normal $\boldsymbol{N}(\boldsymbol{X}_0)$ which surround a narrow band-shaped region Ω_0^k of width k_0 as shown in Fig. 1, whereby it is tacitly assumed that k_0 is much smaller than the extent of Ω_0. As usual, Ω_0^k is interpreted as the localization zone.

With $\boldsymbol{X} = \boldsymbol{X}_0 + \varepsilon\boldsymbol{N}$, we define the natural regularization of $\mathscr{H}_S$ and δ_S as[5]

$$\mathscr{H}_R(\boldsymbol{X}) = \begin{cases} 0 & \text{if } \boldsymbol{X} \in \Omega_0^- \\ \dfrac{1}{2} + \dfrac{\varepsilon}{2k_0} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ 1 & \text{if } \boldsymbol{X} \in \Omega_0^+ \end{cases} \quad \text{and} \quad \delta_R(\boldsymbol{X}) = \begin{cases} \dfrac{1}{k_0} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ 0 & \text{if } \boldsymbol{X} \in \Omega_0^\pm \end{cases} \tag{25}$$

where $\Omega_0^\pm \stackrel{\text{def}}{=} \Omega_0 \backslash \Omega_0^k$. This corresponds simply to the displacement that varies linearly with the coordinate perpendicular to the band. Moreover, δ_S is regularized as

$$\boldsymbol{\delta}_R = \boldsymbol{N}\delta_R = \begin{cases} \dfrac{1}{k_0}\boldsymbol{N} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ 0 & \text{if } \boldsymbol{X} \in \Omega_0^\pm \end{cases} \tag{26}$$

Upon replacing $\mathscr{H}_S$ with $\mathscr{H}_R$ in (24), we obtain the regularized form of the displacement field as

$$\boldsymbol{u} = \boldsymbol{u}_c + [\![\boldsymbol{u}]\!]\mathscr{H}_R \tag{27}$$

The resulting regular deformation gradient is obtained from the definition

$$\boldsymbol{F} \stackrel{\text{def}}{=} \boldsymbol{\varphi} \otimes \boldsymbol{\nabla}_X = \boldsymbol{F}_c + [\![\boldsymbol{u}]\!] \otimes \boldsymbol{\delta}_R \text{ with } \boldsymbol{F}_c = \boldsymbol{\varphi}_c \otimes \boldsymbol{\nabla}_X \tag{28}$$

where $\boldsymbol{F}_c$ is the regularized part of $\boldsymbol{F}$. More explicitly, (28) reads

$$\boldsymbol{F} = \begin{cases} \boldsymbol{F}_c \cdot \left(\boldsymbol{\delta} + \dfrac{1}{k_0}[\![\boldsymbol{U}]\!] \otimes \boldsymbol{N}\right) \text{ with } [\![\boldsymbol{U}]\!] \stackrel{\text{def}}{=} \boldsymbol{F}_c^{-1} \cdot [\![\boldsymbol{u}]\!] & \text{if } \boldsymbol{X} \in \Omega_0^k \\ \boldsymbol{F}_c & \text{if } \boldsymbol{X} \in \Omega_0^\pm \end{cases} \tag{29}$$

Remark: We may interpret $[\![\boldsymbol{U}]\!]$ as the material displacement jump, which is the (contravariant) pullback of $[\![\boldsymbol{u}]\!]$ with the regular portion $\boldsymbol{F}_c$.

The corresponding material rate $\dot{\boldsymbol{F}}$ is obtained from (28) as

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{F}}_c + [\![\dot{\boldsymbol{u}}]\!] \otimes \boldsymbol{\delta}_R = \begin{cases} \dot{\boldsymbol{F}}_c + \dfrac{1}{k_0}[\![\dot{\boldsymbol{u}}]\!] \otimes \boldsymbol{N} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ \dot{\boldsymbol{F}}_c & \text{if } \boldsymbol{X} \in \Omega_0^\mp \end{cases} \tag{30}$$

where it was used that $\boldsymbol{\delta}_R$ does not change with time (since $\boldsymbol{N}$ is a material normal).

[5] Henceforth, we define the notations Ω_0^- and Ω_0^+ such that $\Omega_0 = \Omega_0^+ \bigcup \Omega_0^- - \bigcup \Omega_0^k$.

Next, we compute the regularized spatial velocity gradient $\boldsymbol{l} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$ on the current configuration Ω. Since $\boldsymbol{F}$ is composed of a regular (and invertible) part $\boldsymbol{F}_c$ and a rank-one update, cf (29), it is possible to calculate $\boldsymbol{F}^{-1}$, upon using the Sherman–Morrison formula, to obtain

$$\boldsymbol{F}^{-1} = \begin{cases} \left(\boldsymbol{\delta} - \dfrac{1}{k_0 + \boldsymbol{N} \cdot [\![\boldsymbol{U}]\!]} [\![\boldsymbol{U}]\!] \otimes \boldsymbol{N}\right) \cdot \boldsymbol{F}_c^{-1} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ \boldsymbol{F}_c^{-1} & \text{if } \boldsymbol{X} \in \Omega_0^{\pm} \end{cases} \tag{31}$$

Using (31), we may express the spatial unit normal $\boldsymbol{n}$ of the band, given in (21), as

$$\boldsymbol{n} = j^{-1} \boldsymbol{N} \cdot \boldsymbol{F}^{-1} = \frac{k}{k_0 + \boldsymbol{N} \cdot [\![\boldsymbol{U}]\!]} \boldsymbol{N} \cdot \boldsymbol{F}_c^{-1} \tag{32}$$

where we have also introduced the spatial band thickness k from the relation $jk = k_0$. Now, upon inserting (32) into (31), we may rewrite the expression for $\boldsymbol{F}^{-1}$ as

$$0\boldsymbol{F}^{-1} = \begin{cases} \boldsymbol{F}_c^{-1} \cdot \left(\boldsymbol{\delta} - \dfrac{1}{k} [\![\boldsymbol{u}]\!] \otimes \boldsymbol{n}\right) & \text{if } \boldsymbol{x} \in \Omega_0^k \\ \boldsymbol{F}_c^{-1} & \text{if } \boldsymbol{x} \in \Omega_0^{\pm} \end{cases} \tag{33}$$

Remark: With regard to the expressions for $\boldsymbol{F}$ in (29) and $\boldsymbol{F}^{-1}$ in (33), their irregular parts have the "proper" mixed (two point) structure. The product $[\![\boldsymbol{u}]\!] \otimes \boldsymbol{N}$ appearing in (29) is of Eulerian–Lagrangian type, whereas $[\![\boldsymbol{U}]\!] \otimes \boldsymbol{n}$ in (31) is of the Lagrangian–Eulerian type.

We are now in the position to compute $\boldsymbol{l}$ with the aid of (30) and (31), as

$$\boldsymbol{l} = \begin{cases} \boldsymbol{l}_c + \dfrac{1}{k} [\![\hat{\boldsymbol{u}}^c]\!] \otimes \boldsymbol{n} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ \boldsymbol{l}_c & \text{if } \boldsymbol{X} \in \Omega_0^{\pm} \end{cases} \tag{34}$$

where

$$\boldsymbol{l}_c = \dot{\boldsymbol{F}}_c \cdot \boldsymbol{F}_c^{-1} \quad \text{and} \quad [\![\hat{\boldsymbol{u}}^c]\!] \stackrel{\text{def}}{=} [\![\dot{\boldsymbol{u}}]\!] - \boldsymbol{l}_c \cdot [\![\boldsymbol{u}]\!] \tag{35}$$

Remark: The rate $[\![\hat{\boldsymbol{u}}^c]\!]$ has the structure of a contravariant convective rate w.r.t. the regular part $\boldsymbol{F}_c$, i.e.

$$[\![\boldsymbol{U}]\!] = \boldsymbol{F}_c^{-1} \cdot [\![\boldsymbol{u}]\!] \rightsquigarrow [\![\dot{\boldsymbol{U}}]\!] = \boldsymbol{F}_c^{-1} \cdot [\![\hat{\boldsymbol{u}}^c]\!] \tag{36}$$

4.3. *Onset of localization*

The relations developed so far are valid for an arbitrary loading situation in the postlocalized regime, i.e. when $[\![\boldsymbol{u}]\!] \neq 0$ as well as $[\![\dot{\boldsymbol{u}}]\!] \neq 0$. The particular situation involving onset of localization is defined by the simplified conditions

$$[\![\boldsymbol{u}]\!] = 0, \quad [\![\dot{\boldsymbol{u}}]\!] \neq 0 \rightsquigarrow \boldsymbol{F} = \boldsymbol{F}_c, \quad [\![\hat{\boldsymbol{u}}]\!] = [\![\dot{\boldsymbol{u}}]\!] \tag{37}$$

It thus follows that $\dot{\boldsymbol{F}}$ is the same as in (30), whereas $\boldsymbol{l}$ in (34) now takes the simpler form

$$\boldsymbol{l} = \begin{cases} \boldsymbol{l}_c + \frac{1}{k}[\![\dot{\boldsymbol{u}}]\!] \otimes \boldsymbol{n} & \text{if } \boldsymbol{X} \in \Omega_0^k \\ \boldsymbol{l}_c & \text{if } \boldsymbol{X} \in \Omega_0^{\pm} \end{cases} \tag{38}$$

Remark: The classical localization analysis, e.g. Rudnicki and Rice (1980), Ottosen and Runesson (1991), assumes only a weak discontinuity, i.e. $\boldsymbol{\varphi}$ and $\boldsymbol{F}$ are continuous across Γ_0, while $\dot{\boldsymbol{F}}$ and $\boldsymbol{l}$ are discontinuous with the polarized forms in (30) and (34) inside Ω_0^k

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{F}}_c + \frac{1}{k_0}\boldsymbol{c} \otimes \boldsymbol{N}, \quad \boldsymbol{l} = \boldsymbol{l}_c + \frac{1}{k}\boldsymbol{c} \otimes \boldsymbol{n} \tag{39}$$

where $\boldsymbol{c}$ represents the "strength" of the discontinuity. Although it is possible to interpret $\boldsymbol{c}$ as $[\![\dot{\boldsymbol{u}}]\!]$, no kinematic assumption regarding the velocity field $\boldsymbol{v} = \dot{\boldsymbol{\varphi}}$ is made in the literature. In conclusion, the expressions in (39) are exactly the same as those obtained at the onset of strong discontinuity according to the developments above.

In the following, we shall employ the generic expression for the regularized material and spatial velocity gradients in (39), where $\boldsymbol{c} = [\![\dot{\boldsymbol{u}}]\!]$ in $(39)_1$, while $\boldsymbol{c} = [\![\hat{\boldsymbol{u}}^c]\!]$ in $(39)_2$.

4.4. *General condition for a localized state*

In this section we discuss the general condition that must be satisfied for a (regularized) discontinuity $[\![\boldsymbol{u}]\!]$ to develop across Γ_0. This condition will, for convenience only, be established in the spatial format. For this purpose we introduce the notation $\boldsymbol{X}_+ \in \Omega_0^+$ and $\boldsymbol{X}_- \in \Omega_0^-$ such that

$$\boldsymbol{X}_{\pm} = \boldsymbol{X}_0 \pm \frac{k_0}{2}\boldsymbol{N}(\boldsymbol{X}_0) \text{ for } \boldsymbol{X}_0 \in \Gamma_0 \tag{40}$$

which by pushforward to Ω, via $\boldsymbol{x} = \boldsymbol{\varphi}_c(\boldsymbol{X}, t)$, are mapped to $\boldsymbol{x}_+ \in \Omega^+$ and $\boldsymbol{x}_- \in \Omega^-$, respectively, such that

$$\boldsymbol{x}_{\pm} = \boldsymbol{x}_0 \pm \frac{k}{2}\boldsymbol{n}(\boldsymbol{x}_0) \text{ for } \boldsymbol{x}_0 = \boldsymbol{\varphi}_c(\boldsymbol{X}_0, t) \in \Gamma \tag{41}$$

Because of the continuity of $\boldsymbol{l}_c$, we conclude that it has the same value on both sides of the narrow band Ω^k, i.e.

$$[\![\boldsymbol{l}_c]\!] \stackrel{\text{def}}{=} \boldsymbol{l}_c(\boldsymbol{x}_0) - \boldsymbol{l}_c(\boldsymbol{x}_{\pm}) = 0 \tag{42}$$

Due to equilibrium, the traction and its (convective) rate must be continuous across the band, i.e. we have the fundamental condition

$$[\![\mathring{\bar{\boldsymbol{t}}}]\!] = \mathring{\bar{\boldsymbol{t}}}(\boldsymbol{x}_0) - \mathring{\bar{\boldsymbol{t}}}(\boldsymbol{x}_{\pm}) = 0 \tag{43}$$

where $\mathring{\bar{\boldsymbol{t}}}$ is the Truesdell rate of the total traction.[6] Moreover, we have

$$[\![\bar{\boldsymbol{t}}]\!] = \bar{\boldsymbol{t}}(\boldsymbol{x}_0) - \bar{\boldsymbol{t}}(\boldsymbol{x}_{\pm}) = 0 \tag{44}$$

As the generic tangent relation, we use the relation (16) with (17). For $\boldsymbol{x} = \boldsymbol{x}_{\pm}$, we then obtain

[6] We define $\mathring{\bar{\boldsymbol{t}}} = \boldsymbol{n} \cdot \mathring{\bar{\boldsymbol{\sigma}}} = J^{-1}\boldsymbol{n} \cdot \hat{\boldsymbol{\tau}}^{c^*}$ (corresponding to the relation $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma} = J^{-1}\boldsymbol{n} \cdot \boldsymbol{\tau}$).

$$\mathring{\bar{t}}_{\pm} = J_c^{-1}(\hat{\bar{\tau}}^{c\bullet}_{\pm})^t \cdot n = J_c^{-1} n \cdot \mathscr{E}^u_{2\pm} : d_c + \bar{t}_{\pm} \cdot l^t_c \tag{45}$$

whereas, for $x = x_0$, we obtain

$$\begin{aligned}
\mathring{\bar{t}} &= J^{-1}(\hat{\bar{\tau}}^{c\bullet})^t \cdot n \\
&= J^{-1} n \cdot \mathscr{E}^u_2 : d_c + \bar{t} \cdot l^t_c + \frac{J^{-1}}{k}(n \cdot \mathscr{E}^u_2 \cdot n) \cdot c + \frac{J^{-1}}{k}\bar{\tau}_n c \\
&= \mathring{\bar{t}}_{\pm} + n \cdot [\![J^{-1}\mathscr{E}^u_2]\!] : d_c + [\![\bar{t}]\!] \cdot l^t_c + \frac{J^{-1}}{k} q^u_1 \cdot c \\
&= \mathring{\bar{t}}_{\pm} + n \cdot [\![J^{-1}\mathscr{E}^u_2]\!] : d_c + \frac{J^{-1}}{k} q^u_1 \cdot c \quad \text{with } \bar{\tau}_n = n \cdot \bar{\tau} \cdot n
\end{aligned} \tag{46}$$

where the last equality was obtained upon observing (44), i.e. $[\![\bar{t}]\!] = 0$.

In (46) we have introduced the first spatial undrained localization (or acoustic) tensor q^u_1, which is defined in terms of the second spatial localization tensor q^u_2 as follows:

$$q^u_1 = q^u_2 + \tau_n \delta \text{ with } q^u_2 = q_2 + J K_f n \otimes n \tag{47}$$

Remark: It is of interest to note that the geometrically nonlinear contribution to q^u_1 is expressed solely by the effective normal stress on the discontinuity surface.

For simplicity, we have introduced the notation $\mathring{\bar{t}}_{\pm} \mathring{\bar{t}}(x_{\pm})$ and the convention that $\mathring{\bar{t}} = \mathring{\bar{t}}(x_0)$, i.e. no subscript indicates a quantity evaluated at x_0. Moreover, the jump $[\![J^{-1}\mathscr{E}^u_2]\!]$ is defined as

$$[\![J^{-1}\mathscr{E}^u_2]\!] = J^{-1}(x_0)\mathscr{E}^u_2(x_0) - J_c^{-1}(x_{\pm})\mathscr{E}^u_2(x_{\pm}) \tag{48}$$

From (43) and (46), we may establish the condition

$$\begin{aligned}
\frac{J^{-1}}{k} q^u_1 \cdot c &= -n \cdot [\![J^{-1}\mathscr{E}^u_2]\!] : d_c \\
&= -n \cdot [[\![J^{-1}\mathscr{E}_2]\!] + [\![J]\!] K_f \delta \otimes \delta + [\![Jp]\!] 2I^{sym} - \delta \otimes \delta)] : d_c
\end{aligned} \tag{49}$$

which is the general localization condition that must be satisfied for maintaining the development of a localized deformation mode.

Remark: The tangent stiffness operator $\mathscr{E}_2$ takes on different values at loading (L) and unloading (U), which is determined by $l(x_0)$ inside the band, and $l_c(x_{\pm})$ outside the band. Hence, it is clear that q_2 (inside the band) will assume the value q^{ep}_2 or q^e_2, corresponding to $\mathscr{E}^{ep}_2$ or $\mathscr{E}^e_2$, depending on whether (L) or (U) takes place in the band. (The appropriate loading conditions are discussed below.) Moreover, it is noted that, in general, $[\![J^{-1}\mathscr{E}_2]\!] \neq 0$. Even in the case of, say, unloading inside as well as outside the band, it is still likely that $[\![J^{-1}\mathscr{E}_2]\!] \neq 0$ in the postlocalized regime. This is so, because the deformation state has become different inside as compared to the outside of the band, and $\mathscr{E}_2$ is a function of deformation.

It is of interest to investigate under which circumstances that $J = J_c$, since this situation provides a simplified expression for (49):

$$\frac{1}{k} q^u_1 \cdot c = -n \cdot [\![\mathscr{E}^u_2]\!] : d_c \tag{50}$$

To this end, we reconsider the expressions of F in (29) and F^{-1} in (33) for $\mathbf{x} \in \Omega^k_0$. With the definition $J = \det F$ and $J_c = \det F_c$, we obtain:

$$J = J_c\left(1+\frac{1}{k_0}[\![U]\!]\cdot N\right), \quad J^{-1} = J_c^{-1}\left(1-\frac{1}{k}[\![u]\!]\cdot n\right) \tag{51}$$

It follows that $J = J_c$ in the special case that $[\![U]\!]\cdot N = [\![u]\!]\cdot n = 0$, which designates a proper shear band.

5. CONDITIONS AT THE ONSET OF LOCALIZATION

5.1. *Preliminaries*

The consequeness of the general localization criterion (49) will be investigated with due consideration to the relevant loading condition inside as well as outside Ω^k. Restriction is subsequently made to the onset of localization, in which case the thermodynamic state is continuous. In particular, this means that $\mathscr{E}^e = \mathscr{E}^e_{\pm}$ and $\mathscr{E}^{ep} = \mathscr{E}^{ep}_{\pm}$. It also follows that $J = J_c$ (since $[\![u]\!] = 0$), which infers that $p = p_c$, and (49) simplifies to

$$\frac{1}{k}q_1^u\cdot c = -n\cdot[\![\mathscr{E}_2]\!] : \ d_c \tag{52}$$

It appears that the source term in this equation stems only from the behavior of the contiguous solid. The pertinent loading condition (L) in Ω^k and in $\Omega^{\pm}$ may be expressed, upon combining (11) and (38), as follows:

$$(m : \mathscr{E}^e_2 + 2m\cdot\tau) : d_c + \frac{1}{k}a\cdot c > 0 \text{ when } d = d(x_0) \tag{53}$$

$$(m : \mathscr{E}^e_2 + 2m\cdot\tau) : d_c > 0 \text{ when } d = d_c(x_0) = d_c(x_{\pm}) \tag{54}$$

where the vector $a(n)$ [and the vector $a^*(n)$ for later use] are defined as

$$a = n\cdot\mathscr{E}_2 : m + 2\tau\cdot m, \quad a^* = n\cdot\mathscr{E}_2 : m^* + 2\tau\cdot m^* \tag{55}$$

Clearly, q_1^u will assume the value $q_1^{u,ep}$ or $q_1^{u,e}$, depending on whether (L) or (U) is the relevant mode of loading inside Ω^k. These acoustic tensors are given as

$$q_1^{u,e} = q_2^e + \tau_n\delta + JK_f n\otimes n \tag{56}$$

$$q_1^{u,ep} = q_2^e - \frac{1}{h}a^*\otimes a + \tau_n\delta + JK_f n\otimes n$$

$$= q_1^e - \frac{1}{h}a^*\otimes a + JK_f n\otimes n \text{ with } q_1^e = q_2^e + \tau_n\delta \tag{57}$$

Following Steinmann *et al.* (1996), we may investigate three principally different loading/unloading scenarios inside and outside Ω^k. However, for brevity we consider only the situation that the conditions for plastic loading are met inside as well as outside Ω^k (which is the case of major interest).

5.2. *Hyperelastic–plastic response inside and outside the localization zone*

With $\mathscr{E}_2(x_0) = \mathscr{E}_2(x_+) = \mathscr{E}_2^{ep}$, we obtain $[\![\mathscr{E}_2]\!] = 0$, which together with (52) gives the condition

$$\frac{1}{k}\boldsymbol{q}_1^{\mathrm{u,ep}}\cdot\boldsymbol{c}=0 \tag{58}$$

It appears that the necessary condition for a nontrivial solution $\boldsymbol{c}\neq 0$ is that $\boldsymbol{q}_1^{\mathrm{u,ep}}$ is singular, and this result is (again) quite independent of the width k of the localization zone.

Similar to the case of one-phase solid behavior, the singularity condition can be satisfied, if a certain condition is satisfied in terms of (1) the orientation $\boldsymbol{n}$ of the failure band, (2) the state variables.

To this end, we consider the eigenvalue problem

$$\boldsymbol{q}_1^{\mathrm{u,ep}}\cdot\boldsymbol{z}=\mu\boldsymbol{q}_1^{\mathrm{e}}\cdot\boldsymbol{z} \tag{59}$$

It can be shown, see Runesson *et al.* (1996), that the smallest eigenvalue $\mu^{(1)}$ is given as

$$\mu^{(1)}=1-\frac{1}{2h}\boldsymbol{a}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{a}^*+\frac{JK_{\mathrm{f}}}{2}\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}-D,\quad \boldsymbol{p}_1^{\mathrm{e}}=(\boldsymbol{q}_1^{\mathrm{e}})^{-1} \tag{60}$$

with

$$D=\frac{1}{2}\left[\left(\frac{1}{h}\boldsymbol{a}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{a}^*+JK_{\mathrm{f}}\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}\right)^2-\frac{4JK_{\mathrm{f}}}{h}(\boldsymbol{a}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n})(\boldsymbol{a}^*\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n})\right]^{1/2} \tag{61}$$

The corresponding eigenvector $\boldsymbol{z}^{(1)}$ is given as

$$\boldsymbol{z}^{(1)}=\gamma\boldsymbol{p}_1^{\mathrm{e}}\cdot\left(\frac{JK_{\mathrm{f}}\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{a}^*}{1+JK_{\mathrm{f}}\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}-\mu^{(1)}}\boldsymbol{n}-\boldsymbol{a}^*\right) \tag{62}$$

where γ is an arbitrary scalar.

Singularity of $\boldsymbol{q}_1^{\mathrm{u,ep}}$, i.e. $\mu^{(1)}=0$, can be expressed as the localization criterion (in terms of the state):

$$\mu\overset{\mathrm{def}}{=}1-\frac{Y}{h}\ \text{or}\ \bar{H}=Y-\boldsymbol{m}^{\mathrm{s}}:\mathscr{E}_2^{\mathrm{e}}:\boldsymbol{m}^{*\mathrm{s}}-2\,\mathrm{tr}(\boldsymbol{m}\cdot\boldsymbol{\tau}\cdot\boldsymbol{m}^{*\mathrm{s}}) \tag{63}$$

where

$$Y=\boldsymbol{a}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{a}^*-\frac{\psi(\boldsymbol{a}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n})(\boldsymbol{a}^*\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n})}{\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}},\quad \psi=\frac{JK_{\mathrm{f}}\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}}{1+JK_{\mathrm{f}}\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}} \tag{64}$$

The eigenvector $\boldsymbol{z}^{(1)}$ is now given as

$$\boldsymbol{z}^{(1)}=\gamma\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{w}\ \text{with}\ \boldsymbol{w}=\psi\frac{\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{a}^*}{\boldsymbol{n}\cdot\boldsymbol{p}_1^{\mathrm{e}}\cdot\boldsymbol{n}}\boldsymbol{n}-\boldsymbol{a}^* \tag{65}$$

6. EXPLICIT LOCALIZATION RESULTS FOR ISOTROPIC MODELS

6.1. *Critical conditions for localization—General*

We shall assume (similar to the geometrically linear theory) that the hardening modulus H decreases monotonically from the onset of yielding with a typical loading parameter. The earliest possibility for localization is then defined by

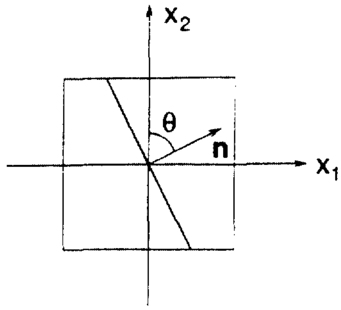

Fig. 2. Orientation of the localization zone in principle stress coordinates.

$$\boldsymbol{n}_{\mathrm{cr}} = \arg[\max(Y(\boldsymbol{n}))] \tag{66}$$

where $Y(\boldsymbol{n})$ was given in $(64)_1$. The critical direction $\boldsymbol{n}_{\mathrm{cr}}$ corresponds to the critical (maximum) value H_{cr} satisfying

$$H_{\mathrm{cr}} = Y(\boldsymbol{n}_{\mathrm{cr}}) - \boldsymbol{m} : \mathscr{E}_2^{\mathrm{e}} : \boldsymbol{m}^* - 2\,\mathrm{tr}(\boldsymbol{m}\cdot\boldsymbol{\tau}\cdot\boldsymbol{m}^*) \tag{67}$$

Because of the assumption of complete elastic and plastic isotropy, the flow direction $\boldsymbol{m}^*$ and the normal to the yield surface $\boldsymbol{m}$ are symmetric tensors, which both commute with $\boldsymbol{\tau}$ and $\boldsymbol{v}^{\mathrm{e}}$ (or $\boldsymbol{b}^{\mathrm{e}}$).

In the numerical examples presented below, we consider only plane strain problems. It is then convenient to represent the critical direction in the plane of interest by θ_{cr}, which is the angle from the (Eulerian) axis corresponding to the smallest principal stress (or elastic stretch) to the normal $\boldsymbol{n}_{\mathrm{cr}}$, as shown in Fig. 2. We shall use the convention that x_1 and x_2 are the in-plane coordinates associated with $\tau_1 \geqslant \tau_2$, whereas x_3 is the out-of-plane direction. Hence, we have

$$\tan^2\theta_{\mathrm{cr}} = \frac{n_1^2}{n_2^2} \tag{68}$$

6.2. *Neo-Hooke hyperelastic model*

As a generic model of hyperelasticity, we choose the Neo-Hooke model defined by the stress decomposition[7]

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\mathrm{iso}} + \boldsymbol{\tau}_{\mathrm{vol}}, \quad \text{with } \boldsymbol{\tau}_{\mathrm{iso}} = G\left(\bar{\boldsymbol{b}}^{\mathrm{e}} - \frac{\bar{I}_1}{3}\boldsymbol{\delta}\right), \quad \boldsymbol{\tau}_{\mathrm{vol}} = \frac{1}{2}K_{\mathrm{b}}((J^{\mathrm{e}})^2 - 1)\boldsymbol{\delta} \tag{69}$$

We have introduced the elastic shear and bulk moduli G and K_{b}, respectively (as pertinent to the small strain limit). Moreover $\bar{I}_1 = \mathrm{tr}\,\boldsymbol{b}^{\mathrm{e}}$ and $J^{\mathrm{e}} = \lambda_1^{\mathrm{e}}\lambda_2^{\mathrm{e}}\lambda_3^{\mathrm{e}}$ and $\bar{\boldsymbol{b}}^{\mathrm{e}}$ is the isochoric part of the left stretch tensor $\boldsymbol{b}^{\mathrm{e}}$.

The corresponding spatial tangent operator is given as

$$\mathscr{E}_2^{\mathrm{e}} = \mathscr{E}_{2,\mathrm{iso}}^{\mathrm{e}} + \mathscr{E}_{2,\mathrm{vol}}^{\mathrm{e}} \tag{70}$$

where

[7] The particular expression for the volumetric part chosen here was originally suggested by Simo and Miehe (1992).

$$\mathscr{E}^{e}_{2,\mathrm{iso}} = \tfrac{2}{3}G\tilde{I}_1 I^{\mathrm{sym}}_{\mathrm{dev}} - \tfrac{2}{3}(\boldsymbol{\delta} \otimes \boldsymbol{\tau}_{\mathrm{iso}} + \boldsymbol{\tau}_{\mathrm{iso}} \otimes \boldsymbol{\delta}) \tag{71}$$

$$\mathscr{E}^{e}_{2,\mathrm{vol}} = K_{\mathrm{b}}(J^{e})^2 \boldsymbol{\delta} \otimes \boldsymbol{\delta} - K_{\mathrm{b}}((J^{e})^2 - 1) I^{\mathrm{sym}} \tag{72}$$

where $I^{\mathrm{sym}}_{\mathrm{dev}} = I^{\mathrm{sym}} - \frac{1}{3}\boldsymbol{\delta} \otimes \boldsymbol{\delta}$. This model gives, with $(57)_2$, the expression

$$\boldsymbol{q}^{e}_1 = G\alpha_1[\boldsymbol{\delta} + \alpha_2 \boldsymbol{n} \otimes \boldsymbol{n} + \alpha_3(\boldsymbol{n} \otimes \boldsymbol{t}'_{\mathrm{iso}} + \boldsymbol{t}'_{\mathrm{iso}} \otimes \boldsymbol{n})] \tag{73}$$

where we have introduced the coefficients

$$\alpha_1(\boldsymbol{n}) = \tfrac{1}{3}\tilde{I}'_1 - \tfrac{1}{2}K'_{\mathrm{b}}((J^{e})^2 - 1) + \tau'_{\mathrm{n}}, \quad \tau'_{\mathrm{n}} = \boldsymbol{n} \cdot \boldsymbol{\tau}' \cdot \boldsymbol{n}$$

$$\alpha_2(\boldsymbol{n}) = \frac{1}{\alpha_1(\boldsymbol{n})}\left[\frac{1}{9}\tilde{I}'_1 + \frac{1}{2}K'_{\mathrm{b}}((J^{e})^2 + 1)\right]$$

$$\alpha_3(\boldsymbol{n}) = -\frac{1}{\alpha_1(\boldsymbol{n})}\frac{2J^{e}}{3} \tag{74}$$

Moreover, we have introduced the nondimensional quantities by scaling with G as follows:

$$\boldsymbol{\tau}' = \frac{\boldsymbol{\tau}}{G}, \quad K'_{\mathrm{b}} = \frac{K_{\mathrm{b}}}{G}, \quad \boldsymbol{t}' = \frac{\boldsymbol{t}}{G} \text{ where } \boldsymbol{t} = \boldsymbol{\tau} \cdot \boldsymbol{n} \tag{75}$$

Using the Sherman–Morisson formula, we may now compute $\boldsymbol{p}^{e}_1 \overset{\mathrm{def}}{=} (\boldsymbol{q}^{e}_2)^{-1}$ in closed form as

$$\boldsymbol{p}^{e}_1 = \frac{\beta_1}{G}[\boldsymbol{\delta} + \beta_2 \boldsymbol{n} \otimes \boldsymbol{n} + \beta_3(\boldsymbol{n} \otimes \boldsymbol{t}'_{\mathrm{iso}} + \boldsymbol{t}'_{\mathrm{iso}} \otimes \boldsymbol{n}) + \beta_4 \boldsymbol{t}'_{\mathrm{iso}} \otimes \boldsymbol{t}'_{\mathrm{iso}}] \tag{76}$$

where

$$\beta_1 = \alpha_1^{-1}$$

$$\beta_2 = -\frac{1}{r}(\alpha_2 - \alpha_3^2|\boldsymbol{t}'_{\mathrm{iso}}|^2)$$

$$\beta_3 = -\frac{\alpha_3}{r}(1 + \alpha_3 \boldsymbol{n} \cdot \boldsymbol{t}'_{\mathrm{iso}})$$

$$\beta_4 = \frac{\alpha_3^2}{r} \tag{77}$$

with

$$r = 1 + \alpha_2 + 2\alpha_3 \boldsymbol{n} \cdot \boldsymbol{t}'_{\mathrm{iso}} + \alpha_3^2(\boldsymbol{n} \cdot \boldsymbol{t}'_{\mathrm{iso}})^2 - \alpha_3^2|\boldsymbol{t}'_{\mathrm{iso}}|^2 \tag{78}$$

Moreover, we need

$$\boldsymbol{a} \overset{\mathrm{def}}{=} \boldsymbol{n} \cdot (\mathscr{E}^{e}_2 : \boldsymbol{m} + 2\boldsymbol{\tau} \cdot \boldsymbol{m}) = 2G(\boldsymbol{f} \cdot \boldsymbol{n} + g\boldsymbol{n}) \tag{79}$$

where

$$f = \gamma_1 \boldsymbol{m} - \tfrac{1}{3} m_{\text{vol}} \boldsymbol{\tau}'_{\text{iso}} + \boldsymbol{\tau}' \cdot \boldsymbol{m}, \quad g = \gamma_2 m_{\text{vol}} - \tfrac{1}{3} \boldsymbol{\tau}'_{\text{iso}} : \boldsymbol{m} \tag{80}$$

with

$$\gamma_1 = \frac{\tilde{I}_1}{3} - \frac{1}{2} K'_{\text{b}}((J^{\text{e}})^2 - 1) = \alpha_1(\boldsymbol{n}) - \tau'_{\text{n}}$$
$$\gamma_2 = \frac{1}{2} K'_{\text{b}}(J^{\text{e}})^2 - \frac{\tilde{I}_1}{9} \tag{81}$$

In the same fashion we obtain

$$\boldsymbol{a}^* \overset{\text{def}}{=} \boldsymbol{n} \cdot (\mathscr{E}_2^{\text{e}} : \boldsymbol{m}^* + 2\boldsymbol{\tau} \cdot \boldsymbol{m}^*) = 2G(\boldsymbol{f}^* \cdot \boldsymbol{n} + g^* \boldsymbol{n}) \tag{82}$$

where

$$\boldsymbol{f}^* = \gamma_1 \boldsymbol{m}^* - \tfrac{1}{3} m^*_{\text{vol}} \boldsymbol{\tau}'_{\text{iso}} + \boldsymbol{\tau}' \cdot \boldsymbol{m}^*, \quad g^* = \gamma_2 m^*_{\text{vol}} - \tfrac{1}{3} \boldsymbol{\tau}'_{\text{iso}} : \boldsymbol{m}^* \tag{83}$$

where the scalars γ_1 and γ_2 are the same as in (81).

We are now in the position to calculate

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{p}_1^{\text{e}} \cdot \boldsymbol{a}^* &= 4G\beta_1 [\boldsymbol{n} \cdot \boldsymbol{f} \cdot \boldsymbol{f}^* \cdot \boldsymbol{n} + \beta_2 (\boldsymbol{n} \cdot \boldsymbol{f} \cdot \boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{f}^* \cdot \boldsymbol{n}) \\
&\quad + \beta_3 (\boldsymbol{n} \cdot \boldsymbol{f} \cdot \boldsymbol{n} + g)(\boldsymbol{t}'_{\text{iso}} \cdot \boldsymbol{f}^* \cdot \boldsymbol{n}) + \beta_3 (\boldsymbol{n} \cdot \boldsymbol{f}^* \cdot \boldsymbol{n} + g^*)(\boldsymbol{t}'_{\text{iso}} \cdot \boldsymbol{f} \cdot \boldsymbol{n}) \\
&\quad + \beta_4 g (\boldsymbol{n} \cdot \boldsymbol{f}^* \cdot \boldsymbol{n}) + \beta_4 g^* (\boldsymbol{n} \cdot \boldsymbol{f} \cdot \boldsymbol{n}) + \beta_5 g g^*] \\
\boldsymbol{a} \cdot \boldsymbol{p}_1^{\text{e}} \cdot \boldsymbol{n} &= 2G\beta_1 [\beta_4 (1 + \boldsymbol{n} \cdot \boldsymbol{f} \cdot \boldsymbol{n}) + \beta_3 (\boldsymbol{n} \cdot \boldsymbol{f} \cdot \boldsymbol{t}'_{\text{iso}} + g \tau'_{\text{iso,n}})] \\
\boldsymbol{a}^* \cdot \boldsymbol{p}_1^{\text{e}} \cdot \boldsymbol{n} &= 2G\beta_1 [\beta_4 (1 + \boldsymbol{n} \cdot \boldsymbol{f}^* \cdot \boldsymbol{n}) + \beta_3 (\boldsymbol{n} \cdot \boldsymbol{f}^* \cdot \boldsymbol{t}'_{\text{iso}} + g^* \tau'_{\text{iso,n}})] \\
\boldsymbol{n} \cdot \boldsymbol{p}_1^{\text{e}} \cdot \boldsymbol{n} &= \frac{\beta_6}{G}
\end{aligned}$$

and

$$\psi = \frac{\beta_6 J K'_{\text{f}}}{1 + \beta_6 J K'_{\text{f}}} \quad \text{with } K'_{\text{f}} = \frac{K_{\text{f}}}{G} \tag{84}$$

where the additional coefficients β_4, β_5 and β_6 are given as

$$\begin{aligned}
\beta_4 &= 1 + \beta_2 + \beta_3 \tau'_{\text{iso,n}} \\
\beta_5 &= 1 + \beta_2 + 2\beta_3 \tau'_{\text{iso,n}} = \beta_4 + \beta_3 \tau'_{\text{iso,n}} \\
\beta_6 &= \beta_1 \beta_5
\end{aligned} \tag{85}$$

6.3. *Special case*: *Mohr–Coulomb yield criterion*

The yield function Φ and the plastic potential Φ^* (allowing for nonassociative dilatancy) related to the Mohr–Coulomb criterion can be expressed in principal components as

$$\Phi = \tfrac{1}{2}(\tau_{\text{I}} - \tau_{\text{III}}) + \tfrac{1}{2}(\tau_{\text{I}} - \tau_{\text{III}}) \sin\phi - c \tag{86}$$

$$\Phi^* = \tfrac{1}{2}(\tau_{\text{I}} - \tau_{\text{III}}) + \tfrac{1}{2}(\tau_{\text{I}} + \tau_{\text{III}}) \sin\phi^* \tag{87}$$

where $\tau_{\text{I}} \geqslant \tau_{\text{II}} \geqslant \tau_{\text{III}}$ are the principal (Kirchhoff) stresses (which are taken positive in

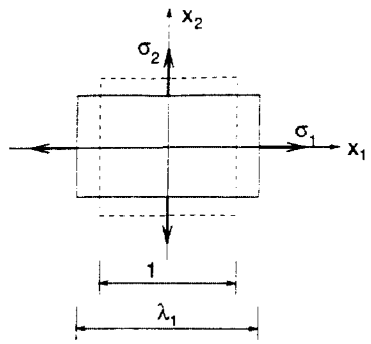

Fig. 3. Biaxial loading in plane strain.

tension), ϕ is the angle of internal friction, ϕ^* is the angle of dilatancy, and c is a cohesion intercept.

Three cases are distinguished depending on which stress component is identified with the intermediate principal stress τ_{II}. (It is recalled that $\tau_1 \geqslant \tau_2$ due to the chosen ordering of stresses in-plane.) The details are omitted here for the sake of brevity.

6.4. *Explicit results for biaxial loading*

The loading and deformation situation under plane strain condition is shown in Fig. 3. Since it is assumed that λ_2 and $\bar{\sigma}_1 = \sigma_1 - p$ are the control (prescribed) variables, this is a situation of mixed stress–strain control. In this particular case, the Neo-Hooke model gives the principal components[8]

$$\bar{\tau}_i = \tau_i - Jp = GJ^{-(2/3)}\left(\lambda_i^2 - \frac{I_1}{3}\right) - J\bar{p}(J) \tag{88}$$

where $J = \lambda_1\lambda_2$, $I_1 = \lambda_1^2 + \lambda_2^2 + 1$ and

$$\bar{p}(J) = -\tfrac{1}{2}K_b(J - J^{-1}) - K_f \ln J = -G[\tfrac{1}{2}K'_b(J - J^{-1}) + K'_f \ln J] \tag{89}$$

Hence, the pertinent equations are

$$\begin{cases} \bar{\tau}_1 = GJ^{-\frac{2}{3}}\left(\lambda_1^2 - \dfrac{I_1}{3}\right) - J\bar{p}(J) = J\bar{\sigma}_1 \\ \bar{\tau}_2 = GJ^{-\frac{2}{3}}\left(\lambda_2^2 - \dfrac{I_1}{3}\right) - J\bar{p}(J) \\ \bar{\tau}_3 = GJ^{-\frac{2}{3}}\left(1 - \dfrac{I_1}{3}\right) - J\bar{p}(J) \end{cases} \tag{90}$$

It appears that, for a fixed value[9] of $\bar{\sigma}_1$ and given λ_2, we may use $(90)_1$ to solve for λ_1 (by iteration). We may then compute $\bar{\tau}_2$ and $\bar{\tau}_3$ from $(90)_2$ and $(90)_3$, whereafter $\bar{\sigma}_i$ can be calculated as $\bar{\sigma}_i = J^{-1}\bar{\tau}_i$.

In order to investigate the influence of the fluid compressibility on the results, we define the "undrained Poisson's ratio" (at small deformations) as

[8] Superscript e for "elastic" is dropped subsequently, since we shall only be concerned with localization at the onset of yielding.

[9] Prescribed $\bar{\sigma}_1$ represents given confining fluid pressure (air or water) in a plane strain test.

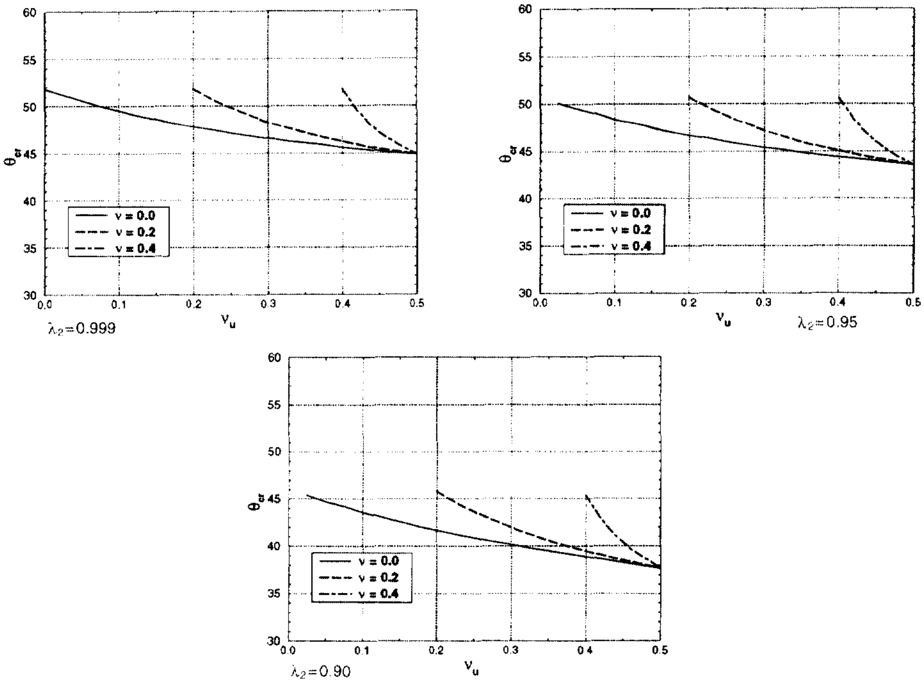

Fig. 4. θ_{cr} vs ν_u for different values of ν and λ_2. Mohr–Coulomb criterion with $\phi = 25.4^\circ$, $\phi^* = 2.8^\circ$. Note: $\theta_{cr} = 45^\circ$.

$$\nu_u = \frac{3(K'_b + K'_f) - 2}{6(K'_b + K'_f) + 2} \tag{91}$$

and it follows simply that $\nu_u = \nu$ when $K'_f = 0$. Hence, for given ν and $\nu_u \in (\nu, 0.5)$, we may solve for K'_b and K'_f from (91).

How θ_{cr} varies with ν_u for given fixed ν is shown in Fig. 4 for different values of λ_2, when $\bar{\sigma}_1 = 0$, at the onset of yielding.[10] The case $\bar{\sigma}_1 = 0$ may be termed unconfined plane strain. Material data are chosen as $\phi = 25.4^\circ$, $\phi^* = 2.8^\circ$ in order to allow a comparison with the closed-form solutions for geometrically linear theory given by Runesson *et al.* (1996). In particular it is confirmed that $\theta_{cr} \rightsquigarrow 45^\circ$ when $\nu_u \rightsquigarrow 0.5$ for small compressive strain ($\lambda_2 = 0.999$). Figure 5, finally, shows how θ_{cr} varies with the (elastic) stretch for $\nu = 0.3$. It appears quite consistently that the small strain value of θ_{cr} is an upper limit to the actual value. In particular, for incompressible response ($\nu_u = 0.5$), values of θ_{cr} down to 40° are possible for highly elastic soft geomaterials.

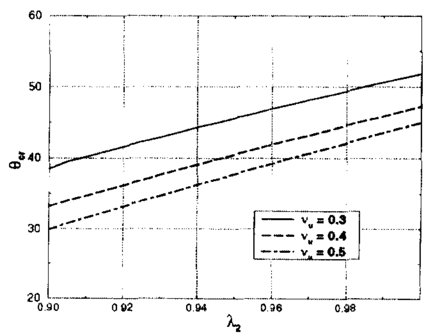

Fig. 5. Critical direction vs stretch at the localization state for $\nu = 0.3$. Note: $\lambda_2 = 1$ represents small strain result.

[10] θ_{cr} is the actual orientation at the onset of localization only if the critical state condition (67) is satisfied.

7. CONCLUDING REMARKS

In this paper we have established the general localization conditions pertinent to undrained response in a hyperelastic–plastic porous solid. Because of the large strain effect, it is not possible in the general case to obtain closed-form expressions for the critical orientation of the localization band. However, the numerical results confirm the small strain results obtained by Runesson *et al.* (1996), in particular that $\theta_{cr} = 45^\circ$ for $\nu_u = 0.5$ independently on the stress state. It is expected that the obtained results are employed to diagnose a particular state as an integral part of a FE-algorithm to capture the evolving localization band. It is conceivable that these findings might have relevance to weak and soft geomaterials under low effective stresses, e.g. peat, dry snow, wet clays and very loose sands. They may also be of interest in the reservoir engineering (in conjunction with high reservoir pressure) and in cases of explosive loading (cavity perforation). These are the situations for which the considered extreme case of undrained shear banding in the presence of compressible fluid could be approached.

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